

Anomalous scaling of conductance cumulants in one dimensional Anderson localization

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(Dated: February 2, 2008)

Abstract

The mean and the variance of the logarithm of the conductance ($\ln g$) in the localized regime in the one-dimensional Anderson model are calculated analytically for weak disorder, starting from the recursion relations for the complex reflection- and transmission amplitudes. The exact recursion relation for the reflection amplitudes is approximated by improved Born approximation forms which ensure that averaged reflection coefficients tend asymptotically to unity in the localized regime, for chain lengths $L = Na \rightarrow \infty$. In contrast the familiar Born approximation of perturbation theory would not be adapted for the localized regime since it constrains the reflection coefficient to be less than one. The proper behaviour of the reflection coefficient (and of other related reflection parameters) is responsible for various anomalies in the cumulants of $\ln g$, in particular for the well-known band center anomaly of the localization length. While a simple improved Born approximation is sufficient for studying cumulants at a generic band energy, we find that a generalized improved Born approximation is necessary to account satisfactorily for numerical results for the band center anomaly in the mean of $\ln g$. For the variance of $\ln g$ at the band center, we reveal the existence of a weak anomalous quadratic term proportional to L^2 , besides the previously found anomaly in the linear term. At a generic band energy the variance of $\ln g$ is found to be linear in L and is given by twice the mean, up to higher order corrections which are calculated. We also exhibit the L -independent offset terms in the variance, which strongly depend on reflection anomalies.

PACS numbers: 71.55.Jv, 72.15.Rn, 05.40.-a

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I. INTRODUCTION

The advent of the scaling theory of localization in d -dimensional disordered systems [1] and more detailed developments of it in 1D [2, 3] and for quasi 1D systems [4] has inaugurated a golden age for mesoscopic physics, particularly the study of transport phenomena.

The fundamental hypothesis in the scaling theory [1] is that the scaling of the logarithm of a typical conductance g as a function of a characteristic size L of the system is described asymptotically for large L by a universal function, $\beta(\ln g)$, of a single parameter (SPS), namely $\ln g$ itself, such that $d \ln g / d \ln L = \beta(\ln g)$. The function β which may generally depend on dimensionality is independent of L and of microscopic parameters in the system. We recall that in the studies of scaling in 1D systems [2, 3] the parameter $\ln(1 + \rho)$, with $\rho = \frac{1}{g}$ the resistance, was identified as the convenient scaling variable both in the localized regime ($\rho \gg 1$) where it is self-averaging and in the low resistance ($\rho \ll 1$) (quasi-metallic) regime. This variable reduces to $-\ln g$ above for $\rho \gg 1$ (localized regime) and, thanks to the Landauer formula, $\rho = \frac{r_L}{t_L}$ (with r_L and t_L the reflection and transmission coefficients of the system, respectively), it coincides with $-\ln t_L \simeq -\ln g$.

In [2, 3] it was argued that the scaling theory of Abrahams et al. [1] had to be interpreted in terms of the scaling of the distribution $P_g(g)$ of the random conductance of the system. SPS means that $P_g(g)$ is fully determined by a single parameter such as e.g. the mean logarithm, $\langle \ln g \rangle$, which is itself defined by a scaling equation of the above form with $\ln g$ replaced by $\langle \ln g \rangle$.

After years of debates the question of the validity of SPS in the theory of Abrahams *et al.* [1] remains still open and has recently been revived [5, 6, 7, 8, 9]. In particular, the justification of the SPS hypothesis in the analyses [2, 3] rests on a random phase approximation (RPA) which assumes that the phases of the amplitude reflection- and transmission coefficients R_L and T_L (with $r_L = |R_L|^2, t_L = |T_L|^2$) are uniformly distributed over $(0, 2\pi)$ in the localization domain, i.e. for length scales L much larger than the localization length ξ . Despite the existence of strong evidence, both numerical [10] and analytical [11, 12] for uniform phase distributions for $L \gg \xi$ in the 1D Anderson model [13] the SPS controversy is not resolved, essentially because phases and conductance are not independent random variables.

Doubts about the validity of results based on RPA have recently led Deych and collab-

orators to reconsider the scaling problem for the exactly soluble Lloyd model [5] as well as to present simulation results for the conductance distribution in the 1D Anderson model in the region of fluctuation states [8]. On the other hand, Schomerus and Titov [6, 7] (see also Roberts [14]) have discussed simulation results for the first four cumulants of $\ln g$ for the Anderson model for weak disorder, both at a generic band energy and at special energies (band center and band edges) where ordinary perturbation theory fails. They also discussed the cumulants using a Fokker-Planck approach for the joint distribution of $\ln g$ and of the transmission phase [6, 7]. Their results support the validity of the lognormal SPS form of the conductance distribution, at a generic band energy (unlike results of Roberts[14]), while showing deviations from SPS at the special energies.

Detailed analytical studies of conductance cumulants based on properties of symmetric groups defined from generalized transfer matrices, and on analytic continuation procedures, have been published earlier by Slevin and Pendry [15] and by Roberts [14]. However, the validation of these approaches rests on support from numerical simulations and studies of limiting cases[16].

In this paper we adopt a new direct approach, circumventing RPA, for studying the conductance distribution (the distribution of $-\ln g = -\ln t_L$) analytically in the localized regime, for weak disorder. An essential ingredient of our analysis is the identification of a general type of anomalous (non-perturbative) effects in various complex reflection amplitude moments and reflection coefficient moments of a finite chain in the localized regime, both for a generic band energy and at the band center. These reflection anomalies strongly influence the logarithmic conductance cumulants. In particular, the anomaly of the second moment of complex reflection amplitudes at the band center is found to be responsible of the well-known Kappus-Wegner anomalies in the localization length [17, 18, 19] and in the variance of the logarithmic conductance [6].

In our analysis we choose to carry out the perturbation theory to fourth order in the disorder (i.e. the random site energies in the Anderson model). Now, by definition we have, for large L , $\langle -\ln g \rangle = \langle -\ln t_L \rangle = 2L/\xi$ (whith $\langle \dots \rangle$ denoting averaging over disorder), where the inverse localization length $1/\xi$ is proportional to the variance of the site energies, $\bar{\varepsilon}^2 = \langle \varepsilon_n^2 \rangle$ ($\langle \varepsilon_n \rangle = 0$), for weak disorder[20]. It follows therefore that in the localized regime ($L \gg \xi$) we have $L \bar{\varepsilon}^2 \gg 1$. Hence it follows that the n -th moment of $-\ln t_L$ is of order $(L \bar{\varepsilon}^2)^n$. This shows that the fourth order perturbation theory may yield a correct

description of the mean and the variance of $-\ln g$ only. While the study of the third- and fourth cumulants, respectively at sixth and eighth orders, is thus left for the future, our results for the first two cumulants at a generic energy already rule out single-parameter scaling if effects beyond leading order ($L\bar{\varepsilon}^2$) are retained.

In Sect. II.A we present the analysis leading to exact formal expressions for the first and second moments of $-\ln g$ in terms of cumulated reflection amplitudes- and reflection coefficients moments. In II.B we define successively the improved Born approximation and a generalized improved Born approximation form of the exact recursion relation relating the random reflection amplitudes of samples of length $n a$ and $(n - 1)a$, respectively. These approximate recursion relations are solved exactly to obtain explicit expressions for reflection amplitudes- and reflection coefficients moments. The generalized improved Born approximation turns out to be vital for studying the band center anomaly in the localization length, in particular for achieving good agreement with the numerical results of Kappus and Wegner and others[6, 14, 16, 17, 18, 19]. In Sect. III we discuss our detailed analytical results for the localization length and for the variance, both at a generic energy and at the band center. Some final remarks follow in Sect. IV.

II. SCALING IN THE ANDERSON MODEL FOR WEAK DISORDER

A. The mean and the variance of $-\ln t_L$

The Schrödinger equation for a chain of N disordered sites $1 \leq m \leq N$ of spacing $a = 1$ ($L = N$) is

$$\varphi_{n+1} + \varphi_{n-1} + \varepsilon_n \varphi_n = E \varphi_n \quad , \quad (1)$$

where the site energies ε_n , in units of a constant hopping rate are mutually independent variables, uniformly distributed between $-\frac{W}{2}$ and $\frac{W}{2}$ ($\langle \varepsilon_m^{2p+1} \rangle = 0$, $\langle \varepsilon_m^2 \rangle = W^2/12$, $\langle \varepsilon_m^4 \rangle = W^4/80$, etc). The disordered chain is connected as usual to semi-infinite non-disordered chains ($\varepsilon_m = 0$) at both ends, with sites $m < 1$, and $m > N$, respectively.

The distribution of the transmission coefficient $t_N = |T_N|^2$ (conductance) for an electron incident from the right with wavenumber $-k$ (energy $E = 2 \cos k$) may be obtained by solving the general recursion relations which connect the complex transmission- (reflection-)

amplitudes $T_n(R_n)$ of a chain of n sites with the corresponding amplitudes for a chain with one less disordered site, of length $n - 1$. These relations derived in [12] are, respectively,

$$T_n = \frac{e^{ik} T_{n-1}}{1 - i\nu_n(1 + e^{2ik} R_{n-1})} \quad , \quad (2)$$

$$R_n = \frac{e^{2ik} R_{n-1} + i\nu_n(1 + e^{2ik} R_{n-1})}{1 - i\nu_n(1 + e^{2ik} R_{n-1})} \quad , \quad (3)$$

with

$$\nu_n = \frac{\varepsilon_n}{2 \sin k} \quad . \quad (4)$$

The fundamental unitarity property,

$$|R_n|^2 + |T_n|^2 = 1 \quad , \quad (5)$$

follows quite generally from (2-3) by expressing $|T_n|^2$, using (2), and $|R_n|^2$, using (3), after rewriting the latter as $R_n = -1 + (1 + e^{2ik} R_{n-1})[1 - i\nu_n(1 + e^{2ik} R_{n-1})]^{-1}$.

From (2) we obtain, with the boundary conditions $T_0 = 1$ and $R_0 = 0$,

$$-\ln t_N = \sum_{n=1}^N \left(\ln[1 - i\nu_n(1 + e^{2ik} R_{n-1})] + c.c. \right) \quad , \quad (6)$$

which, together with (3), is our starting point for studying the probability distribution of $-\ln t_n \simeq -\ln g$ via the calculation of its moments and the corresponding cumulants. We choose to restrict our analysis of the moments $m_j = \langle (-\ln t_N)^j \rangle, j = 1, 2, \dots$, to effects up to 4th order in the random site energies and so we expand (6) in the form

$$-\ln t_N = \sum_{n=1}^N \sum_{p=1}^4 \left(\frac{(-1)^{p+1}}{p} [-i\nu_n(1 + e^{2ik} R_{n-1})]^p + c.c. \right) \quad . \quad (7)$$

As discussed in Sect. I systematic expansion to 4th order permits the explicit study of the first two moments only, in other words the determination of the inverse localization length

$$\frac{1}{\xi} = \frac{m_1}{2N}, N \rightarrow \infty \quad , \quad (8)$$

and of the variance,

$$\text{var}(-\ln t_N) = m_2 - m_1^2, N \rightarrow \infty \quad . \quad (9)$$

After some calculations, using (7), the moments to 4th order in the explicitated ν_n are reduced to the following expressions (with $\bar{\nu}^p = \langle \nu_n^p \rangle$):

$$m_1 = (\bar{\nu}^2 - \frac{\bar{\nu}^4}{2})N + \bar{\nu}^2 \left[(e^{2ik} R_N^{(1)} + \frac{e^{4ik}}{2} R_N^{(2)}) + c.c. \right] , \quad (10)$$

$$m_2 = -\bar{\nu}^2 (e^{4ik} R_N^{(2)} + c.c.) + N\bar{\nu}^4 + N(N-1)(\bar{\nu}^2)^2 \\ + 2(\bar{\nu}^2 - \bar{\nu}^4)Q_N^{(1)} + \frac{\bar{\nu}^4}{2}Q_N^{(2)} , \quad (11)$$

where

$$R_N^{(p)} = \sum_{n=1}^N \langle R_{n-1}^p \rangle , \quad (12)$$

$$Q_N^{(p)} = \sum_{n=1}^N \langle |R_{n-1}|^{2p} \rangle . \quad (13)$$

The factorization of averages in (10-11) results from the fact that the random amplitudes R_{n-1} and R_{n-1}^* are linear functionals depending on ν_1, \dots, ν_{n-1} but not on ν_n , as shown by iterating (3), with $R_0 = 0$. We have used the fact that the odd moments of ν_n are zero, which shows e.g. that $\langle \nu_m \nu_n^p R_{m-1} R_{n-1}^* \rangle = 0, m \neq n, p = 1, 2$. Also, in (10-11) we have systematically dropped all terms of order higher than 4, in particular terms of the form $\langle \nu_m^2 \nu_n^2 (R_{m-1} R_{n-1}^*)^p \rangle, p = 1, 2, m \neq n$, in (11) which are of orders $4 + 2p$, unlike the corresponding terms with $m = n$ which lead to lower order anomalous effects, as shown below.

The Eqs. (10-11) reduce the study of the mean and of the variance of $-\ln t_N$ to the calculation of the quantities $R_N^{(1)}, R_N^{(2)}$ and $Q_N^{(1)}, Q_N^{(2)}$, which we refer to as *fictitious cumulated reflection amplitudes- and cumulated reflection coefficients moments* (sums over chains having lengths equal to rational fractions of Na), respectively. These sums are dominated for $N \rightarrow \infty$ by terms linear in N , reflecting the fact that the amplitudes R_n for $n \rightarrow \infty$ are described by an invariant (stationary) distribution[21, 22] which is independent of the initial site where the iteration of (3) was started. The above cumulated moments will be studied in Sect. III, using the improved Born approximations discussed below.

B. Improved Born approximations

The Born approximation for the random reflection coefficient $|R_n|^2$ for weak disorder is obtained by assuming R_n to be typically proportional to ν_n and approximating (3) by the linear recursion relation $R_N = e^{2ik} R_{N-1} + i\nu_N$, whose solution

$$R_n = i \sum_{m=1}^n e^{2ik(n-m)} \nu_m \quad , \quad (14)$$

yields

$$\langle |R_n|^2 \rangle = n \bar{\nu}^2 = \frac{2n}{\xi_0} \quad , \quad (15)$$

using the familiar perturbation expression for the Anderson localization length[20]. Since the absolute limit of $\langle |R_N|^2 \rangle$ for a chain of length N is unity, it follows from (15) that the Born approximation is not suited for discussing the strong localization (localized) regime, where $N \bar{\nu}^2 \gg 1$ or $N \gg \xi_0$. The same conclusion also follows when using (14) to calculate the second moment of the reflection coefficient which enters in the definition of $Q_N^{(2)}$. In this case, we obtain from (14), $\langle |R_n|^4 \rangle = n \left[\frac{3}{2}(n-1)(\bar{\nu}^2)^2 + \bar{\nu}^4 \right]$, which is also meaningless outside the perturbative domain $(n \bar{\nu}^2)^2 \ll 1$.

In contrast, the study of the localized regime is possible if one uses the improved (first) Born approximation where (3) is approximated, for weak disorder, by

$$R_n = \frac{e^{2ik} R_{n-1} + i \nu_n}{1 - i \nu_n} + \mathcal{O}(\nu_n^2) \quad . \quad (16)$$

For example, by iterating the recursion relation for the averaged reflection coefficient, $\langle |R_n|^2 \rangle$, obtained from (16) (using the fact that R_{n-1} , is independent of ν_n) and summing the resulting geometric series one finds

$$\langle |R_n|^2 \rangle = 1 - (1 - a_1)^n, \quad a_1 = \langle \nu_n^2 (1 + \nu_n^2)^{-1} \rangle \quad , \quad (17)$$

which has the desired limiting value of 1 for $n a_1 \gg 1$, while reducing, to leading order, to the perturbation result (15) in the opposite limit, $n a_1 \ll 1$. It follows that (16) represents the simplest approximation of Eq. (3) which permits a meaningful study of non-perturbative effects in the Anderson model in the localized regime.

On the other hand, from the above discussion it is clear that the expression for the second moment of R_n obtained from (14), namely $\langle R_n^2 \rangle = -e^{4ikn} n \bar{\nu}^2$ is also invalid in the localized regime. In contrast, by solving the recursion relation for $\langle R_n^2 \rangle$ obtained from (16) and summing the corresponding geometric series we get, for any E ,

$$\langle R_n^2 \rangle = \frac{c_2}{e^{4ik}c_1 - 1} + \mathcal{O}(\nu^4), \quad c_{p+1} = \left\langle \frac{\nu_n^{2p}}{(1 - i\nu_n)^2} \right\rangle, \quad p = 0, 1; \quad n\bar{\nu}^2 \gg 1, \quad (18)$$

neglecting exponentially small terms proportional to $\exp(n \ln c_1)$. Here and in the following $\mathcal{O}(\nu^p)$ refers to contributions from terms of p th order in the random site energies. Eq. (18) shows that while to leading order $\langle R_n^2 \rangle$ is proportional to $\bar{\nu}^2$ for $E \neq 0$ (as in perturbation theory), it is strongly enhanced by non-perturbative effects, leading to $\langle R_n^2 \rangle = -\frac{1}{3} + \mathcal{O}(\nu^2)$ at $E = 0$. Using this value one obtains from (10) and (12) (with $\langle R_n^{(1)} \rangle = \mathcal{O}(\nu^2)$) $\frac{1}{\xi} = \frac{\bar{\nu}^2}{3}$, which corresponds to a reduction of 33% of the result $\frac{1}{\xi_0} = \frac{\bar{\nu}^2}{2}$ obtained by perturbation theory[20]. This modification of the inverse localization length at the band center has the same origin as the well-known Kappus-Wegner[17] anomaly. But, clearly, its magnitude calculated within the improved first Born approximation is much too large since the effect obtained by Kappus and Wegner and by others[14, 16, 18, 19], using various sophisticated approaches, range between 8 and 9% of the perturbation result. This leads us to suggest a generalised improved Born approximation of Eq. (3) for dealing specifically with the Kappus-Wegner anomaly, which affects both m_1 , and m_2 at the band center, as shown by (10) and (11). We refer to this more accurate procedure (for the band center) as the generalized improved Born approximation. In this approximation, besides the terms of the improved first Born approximation, we retain the term $i\nu_n e^{2ik} R_{n-1}$ in the numerator of (3) as well as a further term proportional to R_{n-1} obtained by expanding the denominator around the improved form $(1 - i\nu_n)^{-1}$. This yields the approximate recursion relation

$$R_n = e^{2ik} g_n R_{n-1} + f_n, \quad f_n = \frac{i\nu_n}{1 - i\nu_n}, \quad g_n = \frac{1}{(1 - i\nu_n)^2}, \quad (19)$$

where we have ignored a third order term proportional to R_{n-1}^2 . In Sect. 3.2 we return to a crude estimate of the effects of this non-linear term to show that it does not affect the inverse localization length $\frac{1}{\xi}$, nor the variance of $-\ln t_N$, in the localized regime, to leading order in the disorder.

III. LOCALIZATION LENGTH AND VARIANCE OF $-\ln t_N$

We first analyse the moments of $-\ln t_N$ at a generic band energy and then we discuss how the calculations are modified to account for the band center anomalies.

A. Generic band energy

By averaging (16) over disorder (using the fact that ν_n is independent of R_{n-1}) and iterating the resulting recursion relation for $\langle R_n \rangle$ in terms of a geometric series, which is readily summed, we get

$$R_N^{(1)} = \bar{\nu}^2 u (N + u) + \mathcal{O}(\nu^4), u = (e^{2ik} - 1)^{-1} \quad , \quad (20)$$

where we have ignored exponentially small terms proportional to $e^{-N\bar{\nu}^2}$ for weak disorder. Proceeding in a similar way with the equation for $\langle R_n^2 \rangle$ obtained from (16), we find

$$R_N^{(2)} = \bar{\nu}^2 v (N + v) + \mathcal{O}(\nu^4), v = (e^{4ik} - 1)^{-1} \quad . \quad (21)$$

From (10) and (20,21) we then obtain

$$m_1 \equiv \langle -\ln t_N \rangle = \left(\bar{\nu}^2 + \frac{1}{2} [3(\bar{\nu}^2)^2 - \bar{\nu}^4] \right) N - (\bar{\nu}^2)^2 (2|u|^2 + |v|^2) \quad , \quad (22)$$

where the term linear in N yields the familiar fourth order perturbation expression for the inverse localization length ($1/\xi$) at a generic energy [14, 22] and the second term represents a new constant offset of m_1 .

Next we obtain the asymptotic form of the cumulated reflection coefficient moments $Q_N^{(1)}$ and $Q_N^{(2)}$. By summing (17) over the disordered sites we find

$$Q_N^{(1)} = N - a_1^{-1}, Na_1 \rightarrow \infty \quad , \quad (23)$$

up to exponentially small terms. The term $\frac{1}{a_1}$ is the leading deviation of $Q_N^{(1)}$ from the unitarity limit (N) in the localized regime, $Na_1 \sim N\bar{\nu}^2 = \frac{2N}{\xi_0} \gg 1$, in the improved Born approximation.

In order to determine $Q_N^{(2)}$ we have to solve the two-point recursion relation for $\langle |R_{n-1}|^4 \rangle$ derived from (16), namely

$$\langle |R_n|^4 \rangle = b_0 \langle |R_{n-1}|^4 \rangle + b_2 + 4b_1 \langle |R_{n-1}|^2 \rangle - b_1 (e^{4ik} \langle R_{n-1}^2 \rangle + c.c.) \quad , \quad (24)$$

to 4th order non-vanishing terms. Here we have defined

$$b_p = \langle \nu_n^{2p} (1 + \nu_n^2)^{-2} \rangle, p = 0, 1, 2, \dots \quad , \quad (25)$$

and $\langle |R_{n-1}|^2 \rangle$ is given by (17) and $\langle R_{n-1}^2 \rangle = (\nu_n^2(1 - i\nu_n)^{-2})v(1 - e^{4ik(n-1)}) + \mathcal{O}(\nu^4)$. As an example of the form of the solution obtained by iterating such a recursion equation we refer, for brevity's sake, to a similar equation which is solved in Sect. III.B. In the 4th order expression (11) we require $Q_N^{(2)}$ to negative orders in ν up to zeroth order only. After performing successively the summation over sites in the solution of (24) and the further summation over sites in the definition (13) of $Q_N^{(2)}$ we obtain

$$Q_N^{(2)} = \frac{1}{2b_1 + b_2} \left[(4b_1 + b_2) \left(N - \frac{1}{2b_1 + b_2} \right) + 4 \right] - \frac{4}{a_1} \quad , \quad (26)$$

From (9), (11), (21-23) and (25-26) we then obtain, to order ν^4 ,

$$\text{var}(-\ln t_N) = 2\bar{\nu}^2(1 - \bar{\nu}^2)N - \frac{2\bar{\nu}^2}{a_1} + \frac{\bar{\nu}^4(4b_1 + 3b_2)}{2(2b_1 + b_2)^2} + \frac{(\bar{\nu}^2)^2}{1 - \cos 4k} \quad , \quad (27)$$

which again involves a dominant term proportional to N and a higher constant offset term. The leading term $2\bar{\nu}^2 N$ in $\text{var}(-\ln t_N)$ coincides with the result of various earlier theories[2, 6, 14, 15] for a generic band energy. We recall that its proportionality to the dominant term of $\langle -\ln t_N \rangle$ in (22) ensures single-parameter scaling of the lognormal conductance distribution to lowest order, assuming that the higher cumulants are negligible. Our analysis reveals that this basic property is, in fact, a direct consequence of the unitarity limit of the reflection coefficient (17) (which defines $Q_N^{(1)}$ for asymptotic lengths in the localized regime).

On the other hand, the cumulated reflection coefficients moments (23) and (26) are responsible for the existence of non-perturbative constant offset anomalies in the variance (27) in the localized regime, for a generic energy. Finally, we observe that Eqs. (22) and (27) involving terms proportional to N and additional constant offset terms conform to the ansatz of large-deviations statistics for cumulants[7, 23]. To zeroth order, the offset in the variance (27) reduces to the numerical constant -2 . An analogous zeroth order offset constant of value $-\frac{\pi^2}{3}$ has been obtained in[14].

B. Band center

At the band center the denominators in (21) are singular which would require replacing them by their actual form $e^{4ik}c_1 - 1$ in (18) obtained from the improved Born approximation. However, as discussed in Sect. II.B we wish to further improve the calculation of $R_N^{(2)}$ at

the band center by using the generalized improved Born approximation (19) of the exact recursion relation (3).

By squaring (19) and averaging over the disorder we obtain the following relation for determining $\langle R_n^2 \rangle$:

$$\langle R_n^2 \rangle = e^{4ik} C \langle R_{n-1}^2 \rangle + 2e^{2ik} D \langle R_{n-1} \rangle - B \quad , \quad (28)$$

where

$$B = -\langle f_n^2 \rangle, C = \langle g_n^2 \rangle, D = \langle g_n f_n \rangle \quad . \quad (29)$$

On the other hand, by averaging (19), solving for $\langle R_n \rangle$ and performing the summation over sites in the solution we get (with $f = \langle f_n \rangle, g = \langle g_n \rangle$)

$$\langle R_n \rangle = \frac{e^{2ikn} g^n - 1}{e^{2ik} g - 1} f \quad . \quad (30)$$

By inserting (30) in (28), the recursion relation for $\langle R_n^2 \rangle$ takes the form

$$\langle R_n^2 \rangle = e^{4ik} C \langle R_{n-1}^2 \rangle - F(e^{2ik} g)^{n-1} + F - B \quad , \quad (31)$$

where

$$F = 2Df \frac{e^{2ik}}{1 - e^{2ik}g} \quad . \quad (32)$$

The exact solution of (31), with $R_0 = 0$ is given by

$$\langle R_n^2 \rangle = -B G^{n-1} - F \sum_{m=1}^{n-1} G^{n-m-1} (e^{2ik} g)^m + (F - B) \sum_{m=2}^n G^{n-m}, G = e^{4ik} C \quad , \quad (33)$$

which, after performing the geometric sums, reduces to

$$\begin{aligned} \langle R_n^2 \rangle = & -G^{n-1} \left(B + \frac{Fg}{e^{-2ik}G - g} + \frac{B - F}{G - 1} \right) \\ & + (e^{2ik}g)^{n-1} \frac{Fg}{e^{-2ik}G - g} + \frac{B - F}{G - 1} \quad . \end{aligned} \quad (34)$$

Finally, we evaluate $R_N^{(2)}$ defined by (12) and (34), ingoring exponentially small terms proportional to $e^{-N\nu^2}$ (for weak disorder) in the localized regime. Specializing to the band center ($k = \frac{\pi}{2}$) we have

$$R_N^{(2)} = -\frac{1}{C(1-C)} \left(B - \frac{Fg}{C+g} - \frac{B-F}{1-C} \right) + \frac{F}{(C+g)(1+g)} - \frac{N(B-F)}{1-C} \quad , \quad (35)$$

where the anomalous denominator $1-C$ of order $\bar{\nu}^2$ is responsible for the Kappus-Wegner correction in the localization length and a corresponding anomaly in $\text{var}(-\ln t_N)$. By evaluating the quantities entering in (35) and defined in (19), (29) and (32), for weak disorder, we obtain explicitly

$$\begin{aligned} \bar{\nu}^2 R_N^{(2)} &= -\frac{\bar{\nu}^2 N}{10} \left[1 + \frac{\bar{\nu}^4}{2\bar{\nu}^2} + 3\bar{\nu}^2 + \mathcal{O}(\nu^4) \right] \\ &\quad + \frac{1}{100} \left[1 + 3\bar{\nu}^2 + \frac{4\bar{\nu}^4}{\bar{\nu}^2} + \mathcal{O}(\nu^4) \right] \quad . \end{aligned} \quad (36)$$

We recall that this quantity enters with opposite signs in (10) and (11), respectively.

The final expression of $\langle -\ln t_N \rangle$ at the band center is then obtained by substituting (20) and (36) in (10), which yields

$$\begin{aligned} \langle -\ln t_N \rangle &= \frac{N}{10} \left[9\bar{\nu}^2 + 7(\bar{\nu}^2)^2 - \frac{11}{2}\bar{\nu}^4 + \mathcal{O}(\nu^6) \right] \\ &\quad + \frac{1}{100} \left[1 + 3\bar{\nu}^2 + \frac{4\bar{\nu}^4}{\bar{\nu}^2} + \mathcal{O}(\nu^4) \right] \quad . \end{aligned} \quad (37)$$

On the other hand, $\text{var}(-\ln t_N)$ is obtained from (9) and (11) by inserting (23) and (26), and (36-37). This leads to

$$\begin{aligned} \text{var}(-\ln t_N) &= \frac{19}{100} (\bar{\nu}^2)^2 N^2 + \left[2.182\bar{\nu}^2 - 1.02(\bar{\nu}^2)^2 + 0.01\bar{\nu}^4 \right] N \\ &\quad - \frac{1}{100} \left[2.01 + 6.06\bar{\nu}^2 + 8.08 \frac{\bar{\nu}^4}{\bar{\nu}^2} + \mathcal{O}(\nu^4) \right] - \frac{2\bar{\nu}^2}{a_1} + \frac{\bar{\nu}^4}{2} \frac{(4b_1 + 3b_2)}{(2b_1 + b_2)^2} \quad . \end{aligned} \quad (38)$$

The inverse localization length at $E = 0$ obtained from (37),

$$\frac{1}{\xi} = \frac{1}{20} \left[9\bar{\nu}^2 + 7(\bar{\nu}^2)^2 - \frac{11}{2}\bar{\nu}^4 \right] \quad , \quad (39)$$

may be compared to leading order with Thouless' perturbation expression $1/\xi_0 = \bar{\nu}^2/2$ [20]. We thus find that the band center anomaly reduces the numerical coefficient of the $\bar{\nu}^2$ term in $1/\xi$ by 10% with respect to its value in Thouless' expression. For comparison, the earlier

studies of the band center anomaly in the inverse localization length [14, 16, 17, 18, 19] have yielded reductions of the perturbation result ranging between 7.7 and 8.6 %. In view of the simplicity of our analytical treatment leading to a simple transparent picture of the band center anomaly in the localization length, we regard the agreement with the numerical results of the earlier studies as rather satisfactory. Note also the existence of significant band center effects in the coefficients of the quartic terms in (39), as shown by the comparison with the corresponding terms in the perturbation result (Eqs. (8) and (22)) at a generic energy.

The comparison of the variance (38) with the result (27) for a generic energy reveals the existence of two main types of non-perturbative band center anomalies: an enhancement of 9,1 % of the coefficient of the $\bar{\nu}^2 N$ term, on the one hand, and the existence of an additional weak quadratic term proportional to $(N\bar{\nu}^2)^2 \sim (\langle -\ln t_N \rangle)^2$, on the other hand. Such a quadratic term at 4th order is obtained here for the first time. This term is comparable in magnitude with the leading second order term for values of $N\bar{\nu}^2$ of the order of 10 or larger. However, the addition of this term alone does not spoil the single-parameter scaling obtained when restricting to the second order in the disorder. Finally it appears that the accurate description of the Kappus-Wegner anomaly in the localization length, using the generalized improved Born approximation, is crucial for studying the variance since the magnitude of the new N^2 -term in (38) is directly related to this anomaly.

We close this section with a brief remark about the effect of the non-linear term

$$\alpha R_{n-1}^2 \equiv \frac{i\nu_n}{(1 - i\nu_n)^2} e^{4ik} R_{n-1}^2 \quad , \quad (40)$$

which has been omitted on the r.h.s. of (19) in the expansion of (3) for weak disorder. For the purpose of a crude estimate we approximate this term by the linearized form $\alpha \langle R_{n-1} \rangle R_{n-1}$, where $\langle R_{n-1} \rangle$ is given by (30), and solve (19) in the presence of this approximate additional term. Thus we replace g_n in (19) by

$$g'_n = g_n \left(1 - i\nu_n \frac{f}{1 + g} \right) \quad , \quad (41)$$

at the band center. In this approximation the dominant effect of the non-linear term in the moments (10-11) of $-\ln t_N$ arises via the denominator $1 - C = 1 - \langle (g'_n)^2 \rangle$ in (35) which is responsible, in particular, for the Kappus-Wegner anomaly in the localization length. From the expansion of the parameters in (41) for weak disorder it follows that the correction term

in this expression leads to a 4th order correction proportional to $(\bar{\nu}^2)^2$ in $1 - C$, thus leaving the dominant second order term of $1 - C$ unchanged. Our estimate thus shows that the non linear term (40) has no effect on the mean and on the variance of $-\ln t_N$ to leading order in the disorder.

Finally, we note that a more realistic study of the term (40) might be to linearize it in terms of the exact solution of the improved Born relation (16) given by

$$R_n = \sum_{m=1}^n e^{2ik(n-m)} \frac{i\nu_m}{1 - i\nu_m} \prod_{p=m+1}^n \frac{1}{1 - i\nu_p} . \quad (42)$$

However, the analytic solution of (19) in the presence of such a linearized form of (40) is clearly very complicated and will not be discussed.

IV. CONCLUDING REMARKS

In this paper we have discussed a new approach for studying conductance cumulants for weak disorder in the Anderson model in the localized regime. It is based on determining complex reflection amplitudes using linear approximations of the exact non linear recursion relation between the reflection amplitudes R_n and R_{n-1} of disordered chains of lengths na and $(n-1)a$, respectively. These approximate linear relations differ, however, essentially from standard weak disorder (perturbation) expansions in that they correctly account for the asymptotic ($n \rightarrow \infty$) unitarity property of the averaged reflection coefficient $\langle |R_n|^2 \rangle$, in the localized regime, for any strength of a finite disorder.

Our analysis relates various anomalous effects of the disorder in the mean and the variance of $-\ln t_N$ in the localized regime to the above asymptotic behavior of the mean reflection coefficient and/or to corresponding behavior of the second moments of the reflection coefficient and of the complex reflection amplitude (at the band center), respectively. This includes the well-known proportionality of $\text{var}(-\ln t_N)$ to $\langle -\ln t_N \rangle \sim \bar{\nu}^2 N$ (to leading order) and the existence of a leading numerically constant offset in $\text{var}(-\ln t_N)$ at a generic energy. It also includes both the Kappus-Wegner[17] anomaly and a leading constant offset in $\langle -\ln t_N \rangle$ at the band center, as well as similar band center anomalies in $\text{var}(-\ln t_N)$, in particular the existence of a new term proportional to N^2 (which we have obtained for the first time).

An important aspect of our treatment is that it does not rely on the improper use of assumptions about phases such as the phase randomization assumption which has frequently

been invoked in previous work[2, 3].

The results of Sect. 3 indicate that beyond second order in the disorder and ignoring the offset terms, the variance of $-\ln t_N$ cannot be expressed in terms of the mean alone. This rules out single parameter scaling of the distribution of $-\ln t_N$ even if one assumes the higher cumulants to be negligible (in which case the distribution would be lognormal). The third, fourth ... cumulants can, of course, also be studied using our general approach, but, as shown in Sect. 1, this requires perturbation expansions to 6th, 8th ... order in the disorder.

Another application of the analysis of this paper would be the study of conductance cumulants in coupled two- and three-chain systems i.e. for few channel quasi-one dimensional systems. Localization in such systems has recently been discussed for weak disorder both in the case where all the states at the fermi energy belong to conducting bands of the pure few-channel system and in the case where, on the contrary, the states for some of the bands correspond to imaginary wavenumbers (evanescent states) at the fermi energy[24].

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